A NEW IMPLICIT TIME INTEGRATION METHOD FOR NONLINEAR STRUCTURAL DYNAMICS PROBLEMS

Kamil AYDIN

Erciyes University, Department of Civil Engineering, Kayseri, TURKEY

kaydin@erciyes.edu.tr

Abstract: A new time integration method is suggested for solving equation of motion of structural dynamics problems. The method is established upon the principle of impulse-momentum, leading to a lesser number of assumed fields. The algorithmic properties of the procedure are determined by stability and accuracy analyses. Overshooting tendency and order of accuracy are also examined. It is shown that the proposed method is unconditionally stable and non-dissipative. Its numerical dispersion appears to be much less than the commonly used integration methods. The method has no tendency to overshoot both the displacement and velocity response solutions. Its order of accuracy is around four as compared to two of the other methods considered in the study. A few numerical examples consisting of both linear and nonlinear single and multi-degree of freedom systems are carried out to see the overall behavior of the method in various practical problems.

Keywords: Time Integration, Implicit Procedure, Numerical Stability and Accuracy, Overshooting

Introduction

Many methods have been proposed in the last fifty years (Katsikadelis, 2013) for the numerical solution of equations of motion encountered in seismic and structural dynamic problems. These methods can fundamentally be classified as either explicit or implicit methods (Wood, 1990; Chung and Lee, 1994; Bathe, 1996). In explicit integration, the equation of motion of the current time step is not employed in determining the displacement of the current time step. The great advantage of explicit integration is that it does not require solving a set of algebraic equations at each time step (Rio et al., 2005) leading to less computation. Yet, according to the second barrier of Dahlquist (Dahlquist, 1963), all explicit methods are conditionally stable with respect to the size of time step. On the other hand, most implicit integration methods are unconditionally stable. Unconditional stability is an important property to be considered when selecting the proper integration algorithm for analysis of structures with a large number of degrees of freedom. The disadvantage of implicit integration methods is that they require the solution of a system of equations at each time step (Hulbert and Chung, 1996; Chang and Liao, 2005). This makes them computationally more expensive per time step. Apparently, each type of integration has its own advantages and disadvantages. It would be instrumental for an integration method to possess the advantages of explicit and implicit procedures simultaneously. For this purpose, several explicit algorithms with unconditional stability have recently been proposed (Chang and Huang, 2010; Chang, 2002, Chang, 2007; Kolay and Ricles, 2014).

It has been suggested that a desirable time integration method should possess the following criteria (Dokainish and Subbaraj, 1989; Hughes, 1987; Hilber and Hughes, 1978): unconditional stability for application to both linear and nonlinear problems, at least a second order accuracy, self-starting and one step scheme, no more than one set of implicit equations to be solved at each time step, controllable algorithmic damping in higher modes and no overshooting. Many methods have been developed in the last few decades to satisfy these criteria such as the Newmark family methods (Newmark, 1959), Houbolt method (Houbolt, 1950), Wilson-θ method (Bathe and Wilson, 1973), Park method (Park, 1975), HHT-α method (Hilber et al., 1977), WBZ-α method (Wood et al., 1981), generalized-α method (Chung and Hulbert, 1993), and collocation method (Hilber and Hughes, 1978). These algorithms differ from each other mainly with respect to their numerical dissipation and overshooting characteristic. None of them fully satisfies all the criteria. It seems that the development of an effective and accurate integration scheme that satisfies the aforementioned criteria will still be the aim of future research studies in the respective field.

The experience has shown that single-step implicit and unconditionally stable methods are the most preferred methods for the solution of dynamic response of structural systems (Wilson, 2002). Implicit methods use the equation of motion of a dynamic system at the end of each time step to determine the response variables. Direct time integration methods assume a specific type of variation of displacement $u$, velocity $\dot{u}$ and acceleration $\ddot{u}$ within each time increment (Dukkipati, 2009). Instead of equation of motion, equation of impulse-momentum is utilized in this study to relate the unknown parameters. Hence, the proposed method is based upon employing displacement and velocity fields as the unknowns. This results in a decrease in the order of assumed quantities.
Eventually, this decrease is expected to lead to smaller errors in comparison with the existing integration schemes. Numerical stability, dispersion, dissipation are investigated to evaluate the accuracy of the proposed method. Overshooting effect, which is not related to the stability and accuracy characteristics of an integration algorithm, is also examined. Finally, a selection of numerical examples comprising both linear and nonlinear single and multiphase degree of freedom systems are studied in order to readily observe and practically evaluate the characteristics of the proposed scheme.

Proposed Method

The general equation of a viscously damped and linear single degree of freedom dynamical system can be expressed as

\[ m\ddot{u} + cu + ku = p(t) \]  

where \( m \), \( c \) and \( k \) are the mass, damping and stiffness of the system; \( p(t) \) is the externally applied force, and \( u, \dot{u} \) and \( \ddot{u} \) are the displacement, velocity and acceleration response. The numeric integration procedure involves discretization of both the response and excitation into small time increments \( \Delta t \). Therefore, by replacing continuous variable \( t \) by the discrete variable \( t_i \), the differential equation changes to

\[ m\ddot{u}_i + cu_i + ku_i = p_i \]  

This is equation is solved progressively in time increments \( \Delta t \) starting from the known initial conditions. Integration of Eq. (2) yields

\[ m\Delta\dot{u}_i + c\Delta u_i + k\Delta q_i = I_i \]  

where \( I_i \) is the impulse of the applied force and \( \Delta p = p_{i+1} - p_i \). The derivative of acceleration response is assumed to be linear in the proposed study. This yields a second order acceleration and fourth order displacement function as follows:

\[ \ddot{u}(\tau) = \ddot{u}_i + \frac{\tau}{\Delta t} \Delta\ddot{u}_i \]  

\[ \Delta\ddot{u}(\tau) = \ddot{u}_i \tau + \frac{\tau^2}{2\Delta t} \Delta\dddot{u}_i \]  

\[ \Delta u(\tau) = \dot{u}_i \tau + \frac{\dot{u}_i \tau^2}{2} + \frac{\dot{u}_i \tau^3}{6} + \frac{\dot{u}_i \tau^4}{24\Delta t} \Delta\dddot{u}_i \]  

\[ \Delta q(\tau) = u_i \tau + \frac{u_i \tau^2}{2} + \frac{u_i \tau^3}{6} + \frac{u_i \tau^4}{24} + \frac{\tau^5}{\Delta t} \Delta\dddot{u}_i \]  

From Eq. (9), \( \Delta\dddot{u} \) is obtained as

\[ \Delta\dddot{u}_i = \frac{24\Delta u_i}{\Delta t^3} - \frac{24\dot{u}_i}{\Delta t^2} - \frac{12\ddot{u}_i}{\Delta t} - 4\dddot{u}_i \]  

Based on this equation, the expressions for acceleration, velocity and \( q_i \) can be rewritten in incremental form:

\[ \Delta\dddot{u}_i = \frac{12\Delta u_i}{\Delta t^3} - \frac{12\dot{u}_i}{\Delta t} \]  

\[ \Delta\dot{u}_i = \frac{-\dddot{u}_i \Delta t^2}{6} + \frac{4\Delta u_i}{\Delta t} - 4\dot{u}_i - \dddot{u}_i \Delta t \]  

\[ \Delta q_i = \frac{u_i \Delta t + \dddot{u}_i \Delta t^2}{2} + \frac{\dot{u}_i \Delta t^3}{6} + \frac{\dddot{u}_i \Delta t^4}{24} + 24\beta\Delta t\Delta u - 24\dddot{u}_i \beta \Delta t^3 - 12\dddot{u}_i \beta \Delta t^2 - 4\dddot{u}_i \beta \Delta t \]  

Substituting the above equalities in Eq. (4), one determines \( \dddot{u} \)
\[
\ddot{u}_i = \frac{1}{a_i} \left[ \dddot{u}_i \left( \frac{-m\Delta t + \frac{k\Delta t^3}{6} - 12k \beta \Delta t^2}{a_i} \right) + \dddot{u}_i \left( \frac{-4m + \frac{k\Delta t^2}{2} - 24k \beta \Delta t^2}{a_i} \right) + u_i k \Delta t \right] + \Delta u_i \left( \frac{4m}{\Delta t} + c + 24k \beta \Delta t \right) - I_i
\]

with
\[
a_i = \left( -\frac{m\Delta t^2}{6} + \frac{k\Delta t^3}{24} - 4k \beta \Delta t^4 \right)
\]

Once \(\ddot{u}\) has been determined, it can be substituted in Eqs. (12) and (13) to give
\[
\Delta \ddot{u}_i = -\frac{\Delta t}{a_i} \dddot{u}_i \left( \frac{-m\Delta t + \frac{k\Delta t^3}{6} - 12k \beta \Delta t^2}{a} \right) - \Delta t \frac{4m}{\Delta t} + c + 24k \beta \Delta t = \Delta t \frac{4m}{\Delta t} + c + 24k \beta \Delta t
\]

\[
\Delta \ddot{u}_i = \Delta t \frac{4m}{\Delta t} + c + 24k \beta \Delta t
\]

where
\[
I_i = \frac{1}{2}(p_i + p_{i+1}) \Delta t
\]

Substituting Eqs. (17) and (18) into Eq. (9) yields
\[
\begin{align*}
\Delta \ddot{u}_i &= -\frac{\Delta t}{a_i} \dddot{u}_i \left( \frac{-m\Delta t + \frac{k\Delta t^3}{6} - 12k \beta \Delta t^2}{a_i} \right) - \Delta t \frac{4m}{\Delta t} + c + 24k \beta \Delta t = \Delta t \frac{4m}{\Delta t} + c + 24k \beta \Delta t \\
&= \Delta t \frac{4m}{\Delta t} + c + 24k \beta \Delta t
\end{align*}
\]

from which \(\Delta u_i\) value can be determined as
\[
\Delta u_i = \frac{\Delta p}{k_i}
\]

where
\[
k_i = 12 \frac{m}{\Delta t^2} - \frac{m}{a_i} \left( \frac{4m}{\Delta t} + c + 24k \beta \Delta t \right) + \frac{c \Delta t^2}{6a_i} \left( \frac{4m}{\Delta t} + c + 24k \beta \Delta t \right) + k
\]

and
\[
\Delta p_i = \Delta p_i - \dddot{u}_i \left( -6m - \frac{m}{a_i} \left( -m\Delta t + \frac{k \Delta t^3}{6} - 12k \beta \Delta t^2 \right) \right) - c \Delta t - \frac{c \Delta t^2}{6a_i} \left( -m\Delta t + \frac{k \Delta t^3}{6} - 12k \beta \Delta t^2 \right)
\]

\[
-\dddot{u}_i \left( -12 \frac{m}{\Delta t} - \frac{m}{a_i} \left( -4m + \frac{k \Delta t^2}{2} - 24k \beta \Delta t^2 \right) \right) - 4c - \frac{c \Delta t^2}{6a_i} \left( -4m + \frac{k \Delta t^2}{2} - 24k \beta \Delta t^2 \right)
\]

\[
+ u_i \left( \frac{mk \Delta t^2}{a_i} + k \frac{\Delta t^2}{6a_i} \right) - 1 \left( \frac{m \Delta t + c \Delta t^2}{6a_i} \right)
\]
In the above equations, \( k \) represents the effective stiffness and \( \Delta p \) the effective external loading. Once \( \Delta u \) is known, \( \Delta u \) and \( \Delta u \) can be computed from Eqs. (17) and (18), respectively; and the displacement, velocity and acceleration at time \( i+1 \) can be calculated from the following equations:

\[
\begin{align*}
    u_i + \Delta u &= u_{i+1} \\
    \Delta u_i + \dot{u}_i &= \dot{u}_{i+1} \\
    \Delta u_{i+1} + \ddot{u}_i &= \ddot{u}_{i+1}
\end{align*}
\]

Performance of the Proposed Method

Numerical Stability

In a general discussion of the stability of an integration algorithm, it is common practice to apply the algorithm to a single degree of freedom system governed by

\[
\ddot{u} + 2\zeta \omega \dot{u} + \omega^2 u = p(t)
\]

where \( \zeta \), \( \omega \), and \( p \) are the damping ratio, natural cyclic frequency of the system and (modal) external excitation. Most one-step integration methods recursively relate the displacement \( u_{i+1} \) and velocity \( \dot{u}_{i+1} \) at the end of any arbitrary \( i \)th step to the \( u_i \) and \( \dot{u}_i \) at the beginning of that step as

\[
\begin{bmatrix}
    u_{i+1} \\
    \dot{u}_{i+1} \\
    \ddot{u}_{i+1}
\end{bmatrix} =
\begin{bmatrix}
    A_{11}(\Delta t) & A_{12}(\Delta t) & A_{13}(\Delta t) \\
    A_{21}(\Delta t) & A_{22}(\Delta t) & A_{23}(\Delta t) \\
    A_{31}(\Delta t) & A_{32}(\Delta t) & A_{33}(\Delta t)
\end{bmatrix}
\begin{bmatrix}
    u_i \\
    \dot{u}_i \\
    \ddot{u}_i
\end{bmatrix} = [A]
\begin{bmatrix}
    u_i \\
    \dot{u}_i \\
    \ddot{u}_i
\end{bmatrix}
\]

where \([A]\) is referred to as numerical amplification matrix. The algorithmic characteristics of an integration method are determined from the numerical amplification matrix \([A]\). The particular solution of the forced vibration in Eq. (28) is generally omitted in the investigation of stability conditions since any integration method that is unstable under complementary solution will already be unstable under the addition of particular integral.

The amplification matrix for the proposed method is obtained from

\[
\begin{bmatrix}
    k & 0 & m \\
    -\frac{4}{\Delta t} \left( \frac{4m}{\Delta t} + 24k \beta \Delta \right) & 1 & 0 \\
    -\frac{12}{\Delta t} \left( \frac{4m}{\Delta t} + 24k \beta \Delta \right) & 0 & 1
\end{bmatrix}
\begin{bmatrix}
    u_{i+1} \\
    \dot{u}_{i+1} \\
    \ddot{u}_{i+1}
\end{bmatrix} =
\begin{bmatrix}
    0 \\
    0 \\
    0
\end{bmatrix}
\]

\[
\begin{bmatrix}
    0 \\
    0 \\
    0
\end{bmatrix} = [A]
\begin{bmatrix}
    u_i \\
    \dot{u}_i \\
    \ddot{u}_i
\end{bmatrix}
\]

for the case of zero damping. The components of the amplification matrix can be obtained from this equation, but due to shortage of space, they are not provided herein.

The characteristic equation of amplification matrix can be obtained from the following relationship:

\[
[A] - \lambda [I] = 0
\]

in which \( \lambda \) and \([I]\) are eigenvalues of square matrix \([A]\) and unit matrix, respectively. Expansion of Eq. (30) gives

\[
\lambda^3 - 2\alpha_1 \lambda^2 + \alpha_2 \lambda - \alpha_3 = 0
\]

where \( \alpha_1 \) is the half-trace, \( \alpha_2 \) is the sum of principal minors, and \( \alpha_3 \) is the determinant of \([A]\). Solution of Eq. (31) will give three different eigenvalues denoted by \( \lambda_1, \lambda_2 \) and \( \lambda_3 \). The maximum of the eigenvalues is called the spectral radius:

\[
\rho([A]) = \max \{ |\lambda_1|, |\lambda_2|, |\lambda_3| \}
\]

Eq. (31) has three roots whereas the general equation of free vibration for a single degree of freedom system provides two roots. Therefore, the solution of Eq. (31) gives an extra root called the spurious root. The other two roots are referred to as the principal roots. The roots of the proposed method are given as follows:
It is noted that the proposed method produces no spurious root. The stability of an algorithm can be investigated by examining the two roots. A plot of the spectral radius against \( \Delta t/T \) (or \( \omega \Delta t \) in some cases) shows the stability properties of the algorithm. For an integration procedure to have a stable solution the spectral radius should be less than or equal to 1 for all values of \( \Delta t/T \). This condition is satisfied if \( \beta = 1/144 \) for the proposed method. Figure 1 presents the spectral radius of the proposed method along with those of the central difference, Newmark average acceleration, Newmark linear acceleration, and Wilson-\( \theta \) methods. This plot is obtained for \( \zeta = 0 \). It is observed that the central difference method with \( \Delta t > 0.31T \) and linear acceleration method with \( \Delta t > 0.55T \) becomes unstable. The spectral radii for the remaining methods are always less than or equal to 1 for all \( \Delta t/T \); thus, they are unconditionally stable.

It is well known that the condition of stability may be affected by the presence of physical damping in the system. To study this effect, the spectral radii of the mentioned algorithms are determined using a damping ratio of 5%. Figure 2 shows that the damping causes a downward shift in the low-frequency region of vibration modes but does not influence the critical time of instability for the conditionally stable methods. It is observed that the cusp point of Wilson-\( \theta \) method is delayed from \( \Delta t/T = 2.3 \) to \( \Delta t/T = 2.7 \) in the case of physical damping. Hence, it can be said that the inclusion of damping in the numeric integration algorithms makes the stability condition less restrictive.

**Numerical Accuracy**

The accuracy of a time integration procedure is investigated by assessing two types of errors (i.e., numerical dissipation and numerical dispersion) and the order of accuracy of the procedure. Eq. (28) can be rewritten in terms of finite difference equation upon the elimination of velocity and acceleration terms in the equation:

\[
\begin{align*}
\{u_i\} &= \{u_{i+1}\} - \alpha_1 u_t + \alpha_2 u_{i-1} - \alpha_3 u_{i-2} = 0, \quad i \in \{2,3,\ldots,M-1\} \\
\alpha_1 &= \frac{(\Delta t^4 - 192\beta\Delta t^3)\omega_t + (576\beta\Delta t^2 + 16\Delta t^2 - 16\Delta t^2\xi^2)\omega_t^2 - 48}{(96\beta\Delta t^4 - \Delta t^3)\omega_t^4 + (576\beta\Delta t^3 - 8\Delta t^3\xi)\omega_t^3 + (576\beta\Delta t^2 - 8\Delta t^2\xi^2)\omega_t^2 - ((-48)\Delta t\xi)\omega_t - 48} \\
\alpha_2 &= 1 - \frac{\xi(96\Delta t\omega_t - \Delta t^2\omega_t^2(1152\beta - 16))}{\Delta t^4\omega_t^4 - 96\beta\Delta t^3\omega_t^3 + 8\Delta t^3\xi\omega_t^2 - 576\beta\Delta t^2\omega_t^2 - 576\Delta t^2\omega_t^2 - 16\Delta t^2\xi^2 - 48\Delta t\xi\omega_t - 48} \\
\alpha_3 &= 0 \quad (35)
\end{align*}
\]

The spurious root \( \lambda_3 \) is determined to be zero for all the time steps considered in this study. The principal roots are given by

\[
\lambda_{1,2} = a \pm bj = c(\xi \pm j) \quad (37)
\]

which indicates that the principal roots are complex conjugates of each other and are of the form \( a + bj \) and \( a - bj \) with \( j \) being the complex number. In Eq. (37), \( \bar{\Omega} = \bar{\omega} \Delta t \) is referred to as phase of the numeric solution, \( \bar{\xi} \) is the numeric viscous damping, and \( \bar{\omega} \) is the numeric frequency. The numeric frequency value nominally must be equal.
to theoretical frequency $\omega$ but, in general, this condition is not satisfied. The numeric phase and damping coefficient are given in the following based on the principal roots:

$$\Omega = \tan^{-1}\left(\frac{b}{a}\right) \quad (38)$$

$$\xi = -\frac{\ln(\sqrt{a^2 + b^2})}{2\Omega} \quad (39)$$

If the quantity $\sqrt{a^2 + b^2}$ is less than unity, the time integration procedure provides a positive damping; if it is greater than unity, the numerical damping is negative; and it is equal to unity, then there will be no damping in the solution. As an alternative to numeric damping, an amplitude decay (AD) can also be defined as

$$\text{AD} = 1 - \left(\sqrt{a^2 + b^2}\right)^{2\Omega} \quad (40)$$

A second type of error for measuring the relative accuracy of an integration procedure can be provided by

$$\text{PE} = \frac{T - T_{\text{true}}}{T_{\text{true}}} = \frac{\omega \Delta t - \Omega}{\Omega} \quad (41)$$

This equation is expressed with regard to period elongation, where $T = \frac{2\pi}{\omega}$ is the actual period of vibration. The period elongation for the previously-examined integration methods is plotted in Figure 3 as a function of $\Delta t/T$. It is seen that the proposed method will elongate the true period like the Newmark average acceleration, linear acceleration and Wilson-θ methods. But, the amount of numerical dispersion in the proposed method is much less than those of the other methods. From the figure, it is evident that the central difference method would shorten the actual period.

Figure 4 compares the numerical damping ratio of the proposed method to those of the other methods. It is observed that Newmark average acceleration and the proposed method yield almost no damping. The Newmark linear acceleration method produces numerical damping once it has reached the critical value for stability. The Wilson-θ method has a positive numerical damping while the central difference has negative damping. It can be said that the proposed method is non-dissipative. The data in Figure 4 can also be presented in terms of amplitude decay (AD) in Figure 5, from which it is again noted that the Newmark average acceleration and the proposed method show no amplitude decay in displacement response. The Wilson-θ method would decrease the amplitude of actual oscillation gradually as the frequency of oscillation increases. The amount of amplitude decay in the central difference method and Newmark linear acceleration methods decreases without bound once the value of $\Delta t/T$ becomes equal to the critical time step of the respective method. It indicates that the solutions of these two methods would be meaningless after this point on.
Order of Accuracy

The order of convergence is determined by evaluating the local truncation error in displacement response (Kavetski et al., 2004; Razavi et al., 2007; Gholampour and Ghassemieh, 2013). Varying the time step size and keeping the time instant of computation fixed, the displacement error can be determined from the following equation:

\[ |u - u^{\text{exact}}| = \gamma \left( \frac{\Delta t}{T} \right)^\alpha \]  

where \( u \) and \( u^{\text{exact}} \) are the numerical and exact displacement solutions of the system under harmonic excitation, \( \gamma \) and \( \alpha \) are coefficients that would be determined from a regression analysis. \( \alpha \) is referred to as the order of accuracy of the numeric procedure. The results of the regression analysis are shown in Figure 6 in terms of order of accuracy against the ratio of \( \omega_0/\omega \), where \( \omega_0 \) and \( \omega \) are the cyclic frequencies of the external excitation and system, respectively. It is observed that the proposed method has an order of convergence of about 4. Note that the Newmark’s methods has an order of around 2 (Gholampour and Ghassemieh, 2013), the central difference method has an order of slightly higher than 2 (Razavi et al., 2007), and the Wilson-\( \theta \) method has an order of accuracy of 1 (Soroushian et al., 2012). The fourth order convergence of the proposed method means that if the time step size is halved, the error in displacement response will then be sixteen times smaller.

Overshooting effect

This specific phenomenon was first realized by Goudreau and Taylor (1972) as a property of the Wilson-\( \theta \) method. Despite being unconditionally stable, the method showed a tendency to overshoot substantially the displacement and velocity solution during the first few time steps of calculation. This effect is not related to the stability and accuracy of properties of an integration procedure. Therefore, the tendency of an implicit method to overshoot should be considered in the evaluation of an integration method. In order to study the overshooting behavior of a method, the free vibration of a single degree of freedom system can be considered under non-zero values of initial displacement \( u_i \) and/or velocity \( \dot{u}_i \) at the \( i \)th time step (Hilber and Hughes, 1978; Hulbert and Chung, 1994).

Then, the responses \( u_{i+1} \) and \( \dot{u}_{i+1} \) must be calculated at the end of time step as a function of \( \Omega_i = \omega_i \Delta t \). The status of the phase approaching to infinity \( \Omega_i \rightarrow \infty \) provides an indication of tendency to overshoot.

The displacement and velocity responses at the end of any arbitrary time step for the proposed method are obtained upon the elimination of acceleration terms in Eq. (28) as:

\[ u_{i+1} = u_i - \frac{24u_i\Omega_i^3 + (72u_i + 12\Delta t\dot{u}_i)\Omega_i^2 - 144\Delta t\dot{u}_i}{(\Omega_i^2 + 6\Omega_i + 12)^2} \]  

\[ \dot{u}_{i+1} = \frac{12u_i + \Delta t\dot{u}_i}{\Delta t} - \frac{144u_i\Omega_i^3 + (864u_i + 72\Delta t\dot{u}_i)\Omega_i^2 + (1728u_i + 228\Delta t\dot{u}_i)\Omega_i + 1728u_i}{\Delta t(\Omega_i^2 + 6\Omega_i + 12)^2} \]  

In order for the proposed method to avoid overshooting phenomenon, the powers of \( \Omega_i \) in the numerators of Eqs. (43) and (44) should be less than or equal to the powers of \( \Omega_i \) in their denominators (Chung & Hulbert, 1993). The above equations reveal that there is no overshoot in both displacement and velocity responses of the proposed method. The Newmark’s average acceleration method exhibits no overshoot in displacement but it has a tendency...
to overshoot quadratically in the velocity solution (Hughes, 2000). The Wilson-θ method shows overshooting quadratically both in displacement and velocity solutions (Gholampour and Ghassemieh, 2013).

**Numerical Examples**

In order to confirm the numerical properties and evaluate the overall behavior of the proposed method, two examples are formed. The selection of example problems includes nonlinear single and multi-degree of freedom systems. Results of the study are compared to those of the Newmark’s methods, central difference method and Wilson-θ method.

**Example 1: Nonlinear single degree of freedom system**

The second-order nonlinear equation of motion of a dynamic system is given by

$$10u + 5\ddot{u} + 3u^3 + 200u = 500\cos(25t) - 200\sin(10t)$$

which is subjected to initial conditions of $$u(0)=0$$ and $$\dot{u}(0)=0$$. The displacement response for the first five seconds of time history is obtained using the five mentioned integration methods. The response of the Newmark’s linear acceleration method with a time step increment of 0.001 sec is considered to be exact (Chang and Huang, 2010). In the rest of integration procedures a time step of 0.02 sec is utilized. The obtained time histories are plotted in Figure 7. In order to see clearly the performance of the methods, an error metric is defined at each time instant $$t$$ as

$$\text{error} = \left| u' - u'_{\text{exact}} \right|$$

The values of error versus time parameter are shown in Figure 8. It is observed that the errors of the proposed method are the smallest among the considered methods.

**Example 2: Two-story shear building**

In this example, a two-story shear building is considered with initial conditions of $$\{u(0)=[0 \hspace{1cm} 0]^T$$ and $$\{\dot{u}(0)=[0 \hspace{1cm} 0]^T$$. The columns of the system have nonlinear stiffness as described in Figure 9. The beams are assumed to infinitely stiff and the total mass of the system is concentrated at the floor levels. The building is subjected to external forces at floor levels that have two distinct frequencies of oscillation. The system has also damping and the damping coefficients are indicated in the figure. Story displacement responses are determined through the five methods. Again, the solution of the Newmark’s linear acceleration with $$\Delta t=0.001$$ sec is assumed to be “exact”. Figures 10 and 11 show the computed story displacements and Figure 12 displays the error quantities in the displacement response of the second story. It is observed that the proposed method with $$\Delta t=0.02$$ sec closely follows the solution of the Newmark’s linear acceleration with $$\Delta t=0.001$$ sec. The error in the central difference method appears to be the largest, followed by that of the Wilson-θ method.
Conclusions

A new implicit step-by-step integration algorithm is proposed in this study for the solution of nonlinear problems in structural dynamics. The algorithmic characteristics of the proposed method are determined through stability and accuracy analyses. Also, to observe the general behavior of the proposed scheme in various dynamics problems, numerical tests are carried out in comparison with frequently used integration procedures such as the Newmark’s family methods, central difference method and Wilson-θ method. Based on these studies, the following conclusions can be reached:

- With the use of $\beta=1/144$, the proposed method becomes unconditionally stable.
- The inclusion of physical damping in the system does not influence the unconditional stability of the method. However, it makes the stability conditions of the Newmark’s linear acceleration and the central difference methods less restrictive.
- The numerical dispersion error of the proposed method is much smaller in comparison with the other methods considered in the study.
- The proposed method does not generate any numerical damping; hence, it is non-dissipative.
- This conclusion may be put it differently to state that the proposed scheme shows no amplitude decay regardless of the time step size used.
- The order of accuracy is about 4 for the proposed scheme. It is about 2 for Newmark’s family methods and the central difference method, and 1 for the Wilson-θ method.
- Overshooting analysis reveals that the proposed method has no tendency to overshoot both the displacement and velocity solutions.
- The numerical example problems show that the displacement solutions of the proposed algorithm closely follow the exact solution of the Newmark’s linear acceleration method, where a rather small time increment is employed to simulate the true displacement response.

The behavior of the proposed numerical integration scheme in the solution of large and sophisticated structures comprising both very stiff and flexible components and structures that exhibit spurious high mode responses arising due to inaccuracies and inadequacies in the finite element assemblages may be investigated as a further study.
References
